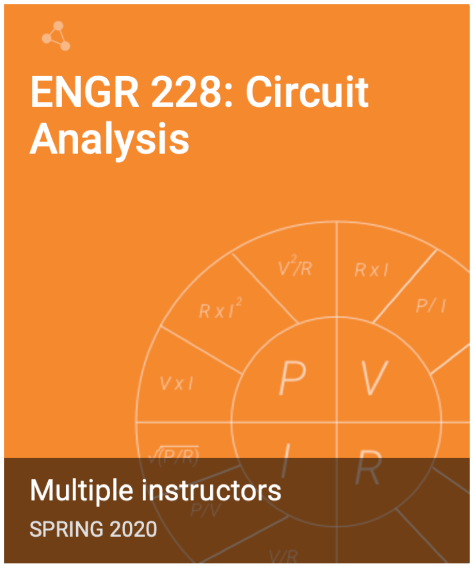


# Chapter 6.8

**Overdamped RLC Circuits**

Engr228 - Circuit Analysis

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## Section 6.8 Objective

- Be able to determine the natural response of overdamped parallel RLC circuits.

## First-Order RL and RC Circuit Review

- Transient, natural, or homogeneous response:
  - Fades over time;
  - Resists change.
- Forced, steady-state, particular response:
  - Follows the input;
  - Independent of time passed.
- The total response will be:

$$i_L(t) = i_L(\infty) + (i_L(0) - i_L(\infty))e^{-\left(\frac{R}{L}\right)t}$$

$$v_c(t) = v_c(\infty) + (v_c(0) - v_c(\infty))e^{-\frac{t}{RC}}$$

## RLC Circuits

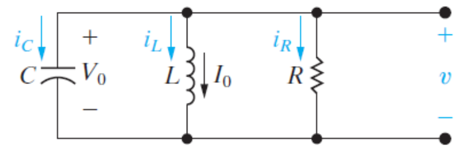
- RLC circuits contain **both** an inductor and a capacitor;
- These circuits have a wide range of applications including oscillators, frequency filters, flight simulation, modeling automobile suspensions, and more;
- The response of RLC circuits with DC sources and switches will consist of a natural response and a forced response:

$$v(t) = v_f(t)+v_n(t)$$

The complete response must satisfy both the **initial conditions** and the **final conditions** of the forced response.

## Source-Free Parallel RLC Circuits

We will first study the natural response of second-order circuit by looking at a source-free parallel RLC circuit:




$$i_R + i_L + i_C = 0$$

$$\frac{v(t)}{R} + \frac{1}{L}\int v(t)dt + C\frac{dv(t)}{dt} = 0$$

$$C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$$

⇨ Second-order Differential equation

## Source-Free Parallel RLC Circuits

This second-order differential equation can be solved by assuming the form of a solution:

$$v(t) = Ae^{st}$$

$$C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$$

$$CAs^2e^{st} + \frac{1}{R}Ase^{st} + \frac{1}{L}Ae^{st} = 0$$

$$Ae^{st}(Cs^2 + \frac{1}{R}s + \frac{1}{L}) = 0$$

which means $Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$

- This is known as the ***characteristic equation***.

## Source-Free Parallel RLC Circuits

Using the quadratic formula, we get

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Define resonant frequency: $\omega_0 = \frac{1}{\sqrt{LC}}$

Define damping factor: $\alpha = \frac{1}{2RC}$
(neper frequency)

Then: $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$

## Second-Order Differential Equation Solution

We will now divide the circuit response into three cases according to the sign of the term under the radical.

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

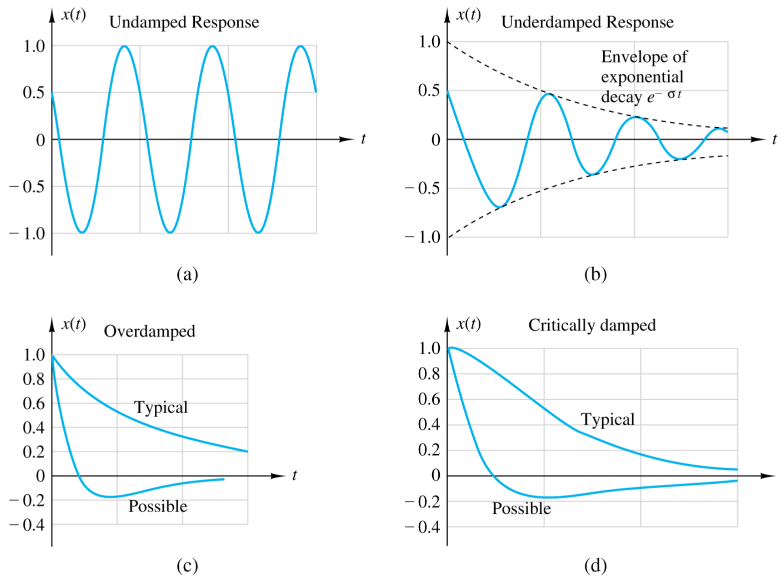
$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$\alpha > \omega_0$ (overdamped): $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

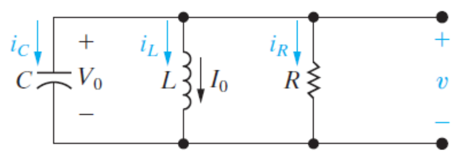
$\alpha = \omega_0$ (critically damped): $v(t) = A_1 t e^{st} + A_2 e^{st}$

$\alpha < \omega_0$ (underdamped): $v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$

# Types of Circuit Responses




# Solving an Overdamped RLC Circuit

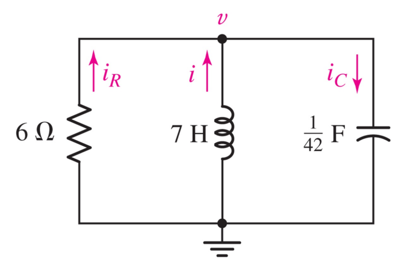



$$v_C(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- We need two equations to solve the second order circuit:
  - Evaluate $v_C$ at $t = 0^+$ $\quad v_C(0^+) = A_1 + A_2$
  - Evaluate $\frac{dv}{dt}$ at $t = 0^+$ $\quad A_1 s_1 + A_2 s_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$
- Note the second equation is equivalent to writing a node equation and evaluating at $t = 0^+$.

## Overdamped Example

Find v(t) in the circuit at the right. Ignore the current arrows.




Given initial conditions:
$v_c(0) = 0$, $i_L(0) = -10A$

$$\alpha = \frac{1}{2RC} = 3.5 \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$\alpha > \omega_0$ therefore this is an overdamped case

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \qquad s_1 = -1, \; s_2 = -6$$

## Overdamped Case - continued

The solution is of the form: $v(t) = A_1 e^{-t} + A_2 e^{-6t}$

Use initial conditions to find $A_1$ and $A_2$

From $v_c(0) = 0$ at t = 0:

$$v(0) = 0 = A_1 e^0 + A_2 e^0 = A_1 + A_2$$

From KCL taken at t = 0:

$$i_R + i_L + i_C = 0$$

$$\frac{v(0)}{R} + (-10) + C\frac{dv(t)}{dt}\bigg|_{t=0} = 0$$

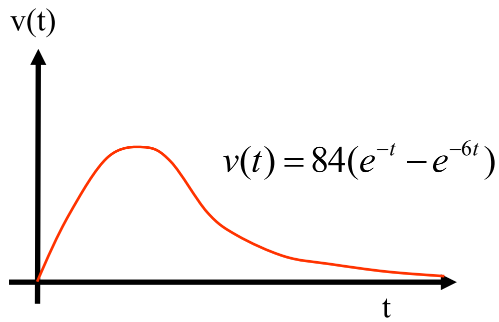
$$\frac{0}{R} + (-10) + \frac{1}{42}\left(-A_1 e^{-t} - 6A_2 e^{-6t}\right)\Big|_{t=0} = 0$$

$$\left(-A_1 - 6A_2\right) = 420$$

## Overdamped Case - continued

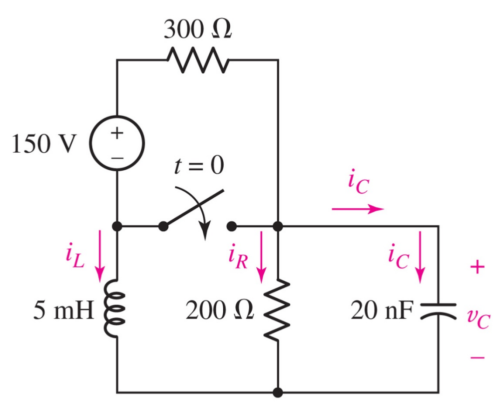
Solving the two equations we get $A_1$ = 84 and $A_2$ = -84
The solution is:

$$v(t) = 84e^{-t} - 84e^{-6t} = 84(e^{-t} - e^{-6t})V$$




## Example: Overdamped RLC Circuit

Find $v_C(t)$ for t > 0.




$v_C(t) = 80e^{-50,000t} - 20e^{-200,000t}$ *V for t > 0*

**Equations for Analysing the Natural Response of Parallel *RLC* Circuits**

| | |
|---|---|
| Characteristic equation | $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$ |
| Neper, resonant, and damped frequencies | $\alpha = \frac{1}{2RC} \quad \omega_0 = \sqrt{\frac{1}{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$ |
| Roots of the characteristic equation | $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$ |
| $\alpha^2 > \omega_0^2$: overdamped | $v(t) = A_1e^{s_1t} + A_2e^{s_2t}, \text{t} \geq 0$ <br> $v(0^+) = A_1 + A_2 = V_0$ <br> $\frac{dv(0^+)}{dt} = s_1A_1 + s_2A_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$ |
| $\alpha^2 < \omega_0^2$: underdamped | $v(t) = B_1e^{-\alpha t}\cos\omega_d t + B_2e^{-\alpha t}\sin\omega_d t, \text{t} \geq 0$ <br> $v(0^+) = B_1 = V_0$ <br> $\frac{dv(0^+)}{dt} = -\alpha B_1 + \omega_d B_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$ |
| $\alpha^2 = \omega_0^2$: critically damped | $v(t) = D_1te^{-\alpha t} + D_2e^{-\alpha t}, \text{t} \geq 0$ <br> $v(0^+) = D_2 = V_0$ <br> $\frac{dv(0^+)}{dt} = D_1 - \alpha D_2 = \frac{1}{C}\left(\frac{-V_0}{R} - I_0\right)$ |

(Note that the equations in the last three rows assume that the reference direction for the current in every component is in the direction of the reference voltage drop across that component.)

# Section 6.8 Summary

- Showed how to determine the natural response of overdamped parallel RLC circuits.